International Journal of Theoretical Physics, Vol. 16, No. 9 (1977), pp. 671–679

Lie Algebras and Dynamic Nonlinear Systems Containing Limit Cycles

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Received: 20 *June* 1977

Abstract

Certain Lie algebras, represented as linear partial differential operators of first order, are used to derive autonomous systems of differential equations which involve limit cycles. To illustrate the approach an example is given.

1. *Introduction*

Dynamic nonlinear systems play an important role in science. We shall assume that the evolution of the system is governed by a real ordinary differential equation, that is, the state $x(t) = (x_1(t), \ldots, x_n(t))$ of the system at time t is a point along the solution of the differential system

$$\dot{x}_i = Y_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n \tag{1.1}$$

which passes through the point $x_i{}^0$ at time $t = t_0$. The dots stand for differentiation with respect to the independent variable t.

In particular, those systems are of interest which contain limit cycles. The importance of limit cycles is that they represent self-sustained oscillations in nonlinear, nonconservative systems. Oscillations of this kind do not depend on the initial conditions of the differential equation.

In the theory of nonlinear electronic networks and other applicatons to physics, limit cycles have been studied by several authors (van der Pol, 1926); Krogdahl, 1955; Blaquiere, 1966; Vojtasek and Janac, 1969; Andronow et al., 1969. The connection of limit cycles and quantum mechanics has been investigated by e Silva et al. (1960). Also in the theory of chemical and biological oscillators, limit cycles are important (Glansdorff and Prigogine, 1971;

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Pavlidis, 1973; Nicolis and Portnow, 1973). From the mathematical point of view interest in this field is mainly focused on three aspects. The first one is the existence of limit cycles for a given differential equation (Levinson and Smith, 1942; Aggarwal, 1972). Another aspect is that of stability (Cesari, 1963; Nemytskii and Stepanov, 1960; Yoshizawa, 1975). A final important question is that of perturbation methods (Hale, 1963; Kruskal, 1962; Kummer, 1971). Note that in most of the works cited only the two-dimensional case has been considered.

In the present paper we consider the problem from another point of view, which has been overlooked so far. We state the connection between a certain class of Lie algebras, represented as linear partial differential operators of first order, and a class of nonlinear autonomous systems of differential equations that contain limit cycles. We study autonomous analytic systems of three coupled first-order differential equations, representing a flow in the three-dimensional space. Note that the extension to higher dimensions is straightforward.

The basic idea of the method described consists of finding for a given vector field X on $M = R^n$ (M = manifold) the most general vector field Y such that

$$[X, Y] = \lambda Y \tag{1.2}$$

$[\ ,\ ]$ denotes the commutator and λ is an arbitrary function. Every vector field has an autonomous system of differential equations associated with it and vice versa. For example, according to equation (1.1) we have

$$Y = \sum_{i=1}^{n} Y_i(x) \frac{\partial}{\partial x_i} \tag{1.3}$$

If we have found the vector field Y, then the limit cycle, assuming it exists, of the system with the vector field Y can be obtained. Thus our problem consists of two parts. The first one is that of finding, for a given vector field X, the most general differential equation invariant under this vector field and, since we are considering autonomous systems, the associated system. As we shall see later, the vector field Y contains n arbitrary functions, where, however, the argument is given by the invariants of the vector field X. Therefore Y represents a class of vector fields (henceforth called vector field for short). In two dimensions the procedure of obtaining the vector field Y has been known for a long time (Cohen, 1911; Eisenhardt, 1961; Campbell, 1966; Bluman and Cole, 1974). It is clear that, in general, this system does not include limit cycles. Hence the second question arises as to how the vector field X (or the corresponding differential equation) must be chosen in order to obtain systems containing limit cycles. In Section 2 we investigate some Lie algebras, represented as linear partial differential operators of first order, on which the starting vector field X is built up. Moreover we reveal how to obtain the vector field Y, which fulfills equation (1.2).

In Section 3 we derive the basic theorem. Section 4 is devoted to a widely discussed example.

2. *Lie Algebras*

Consider the real non-Abelian Lie algebra $so(n)$, denoted by L_1, where the basis is given by the set

$$\left\{x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} : i = 1, \ldots, n-1; j = 2, \ldots, n; i < j\right\} \tag{2.1}$$

As a useful abbreviation we put

$$X_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \tag{2.2}$$

The commutators yield

$$[X_{kl}, X_{ij}] = \delta_{il} X_{kj} + \delta_{ik} X_{jl} + \delta_{jk} X_{li} + \delta_{jl} X_{ik} \tag{2.3}$$

The dimension of the Lie algebra is given by $\dim L_1 = n(n-1)/2$. Note that the Lie algebra L_1 is simple. Thus this Lie algebra is nonsolvable and therefore not nilpotent. As a consequence the center is $C(L_1) = \{0\}$. For $n = 3$ we have the basis X_{12}, X_{13}, X_{23} satisfying

$$[X_{12}, X_{13}] = -X_{23}, \qquad [X_{23}, X_{12}] = -X_{13}, \qquad [X_{13}, X_{23}] = -X_{12} \tag{2.4}$$

Now we introduce the useful vector field

$$Z = \sum_{i=1}^{3} X_{ii} \tag{2.5}$$

where $X_{ii} = x_i \partial/\partial x_i$. Then we find that

$$[Z, X_{ij}] = 0 \tag{2.6}$$

Hence the set $\{Z\} \cup \{L_1\}$ forms a basis of a Lie algebra, denoted by L_2, and it is obvious that the center is given by $C(L_2) = \{0, Z\}$.

It is clear that the set $\{Q_k = \partial/\partial x_k, k = 1, 2, 3\}$ forms a basis of an Abelian Lie algebra. Since

$$[Q_k, X_{ij}] = \delta_{ki} Q_i - \delta_{kj} Q_i \tag{2.7}$$

we find that the set $L_3 = \{Q_k, k = 1, 2, 3\} \cup \{L_2\}$ also forms a basis of a Lie algebra. Finally we observe the useful property that

$$[Q_i - Q_j, X_{ki} + X_{kj}] = 0 \tag{2.8}$$

The Lie algebras L_1, L_2, and L_3 will be used in the following in order to construct vector fields X and Y where the commutator vanishes.

Now we wish to find the invariants of any linear combination of the vector fields belonging to L_1. Invariants of real low-dimensional Lie algebras have been investigated by Patera et al. (1976). The problem of finding invariants will be reduced to that of solving a certain set of linear first-order partial differential equations. These may have polynomial solutions, giving rise to

Casimir operators (lying in the enveloping algebra of the corresponding Lie algebra). They may also have other invariants, i.e., ratios of two polynomials and general invariants (Patera et al., (1976). In the present paper we are only interested in polynomial solutions. Note that for some Lie algebras the corresponding linear first-order partial differential equations have no solutions. In the case under investigation we find that the only invariant is given by $(\Sigma_{i=1}^{n} x_i^2 - c_1)$, where c_1 is a real constant. It follows at once that any linear combination of the basis elements of L_1 has this quantity as an invariant. Here it is possible that further invariants exist. Consider the Lie algebra L_1 with $n = 3$ and the linear combination

$$X = X_{12} + X_{13} \tag{2.9}$$

Then we have as further invariant $x_2 - x_3 + c_2$, where c_2 is a real constant. We are going to determine the vector field Y such that the commutator $[X, Y]$ vanishes, where X is a linear combination of the basis elements of L_1. First of all we consider the rule

$$[X, fZ] = (Xf)Z + f[X, Z] \tag{2.10}$$

If the argument of the function f is given by the invariants of the vector field X, then the first term on the right-hand side is equal to zero.

An additive part of the vector field X can be given at once, namely,

$$Y_1 = f_1 X \tag{2.11}$$

In what follows, the argument of the functions f_1, f_2, and f_3, determined by the invariants of X, is omitted. Considering the Lie algebra L_2 we obtain as the second part

$$Y_2 = f_2 \sum_{i=1}^{3} X_{ii} \tag{2.12}$$

To obtain the third part we must investigate the Lie algebra L_3. For example, let $X = X_{12} + X_{13}$. Then we have

$$[X_{12} + X_{13}, Q_2 - Q_3] = 0 \tag{2.13}$$

3. *Basic Theorem*

The basic idea for obtaining nonlinear systems containing periodic solutions and, as a special case, limit cycles is as follows: Let the real analytic vector fields

$$X = \sum_{i=1}^{3} X_i(x) \frac{\partial}{\partial x_i}, \qquad Y = \sum_{i=1}^{3} Y_i(x) \frac{\partial}{\partial x_i} \tag{3.1}$$

on R^3 be given. We assume that equation (1.2) holds. This leads to

$$0 = \sum_{i=1}^{3} \left[X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right] - \lambda Y_j \tag{3.2}$$

The vector field can be cast into the matrix form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix} \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \tag{3.3}$$

The determinants of the 2 x 2 submatrices on the right-hand side of equation (3.3) have the form

$$X_1 Y_2 - Y_1 X_2, \qquad X_1 Y_3 - Y_1 X_3, \qquad X_2 Y_3 - Y_2 X_3 \tag{3.4}$$

Applying equation (3.2), it follows that

$$Y(X_k Y_l - Y_k X_l) = \sum_{i=1}^{3} (Y_l X_i - X_l Y_i) \frac{\partial Y_k}{\partial x_i} + (X_k Y_i - Y_k X_i) \frac{\partial Y_l}{\partial x_i}$$

(3.5)

$$X(X_k Y_l - Y_k X_l) = \sum_{i=1}^{3} (Y_l X_i - X_l Y_i) \frac{\partial X_k}{\partial x_i} + (X_k Y_i - Y_k X_i) \frac{\partial X_l}{\partial x_i} + \lambda(X_k Y_l - Y_k X_l)$$

Observe that on the right-hand side of equations (3.5) the 2 x 2 submatrices appear described by equations (3.4). Now we investigate the following equations:

$$X_1 Y_2 - Y_1 X_2 = 0, \qquad X_1 Y_3 - Y_1 X_3 = 0, \qquad X_2 Y_3 - Y_2 X_3 = 0 \tag{3.6}$$

Note that the function f_1 does not appear in equations (3.6). It can easily be seen that the equations (3.6) can be fulfilled by inserting the critical points of X. Furthermore, for a certain choice of f_2 and f_3 the equations (3.6) can also be fulfilled. In these cases the invariants of X, appearing as the argument of f_2 and f_3, are chosen so that the intersection of the surfaces, determined by the invariants, leads to a closed curve. The curve obtained is invariant with respect to X and Y. The reason that the curve is closed follows from the fact that X is a linear combination of elements of L_1.

4. *An example*

Consider the vector field

$$X = X_{12} + X_{13} \tag{4.1}$$

with the associated system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4.2}$$

The critical points of X are given by

$$\{x_1 = 0; x_2 = -x_3\} \tag{4.3}$$

With respect to the system of the differential equations the critical points may be interpreted as the time-independent solutions (also called steady state solutions). The eigenvalues of the matrix on the right-hand side of equation (4.2) are found to be $\lambda_{1,2} = \pm i$, $\lambda_3 = 0$. From the foregoing it is evident that the invariants with respect to the vector field X are

$$\{x_1^2 + x_2^2 + x_3^2 - c_1, x_2 - x_3 - c_2\} \tag{4.4}$$

Moreover, as described above too, the vector field X commutes with every part of the vector field

$$Y = f_1 X + f_2 \sum_{i=1}^{3} X_{ii} + f_3(Q_2 - Q_3) \tag{4.5}$$

f_1, f_2, and f_3 are arbitrary functions, where the argument is given by the invariants of the vector field X. The vector field Y leads to the autonomous system

$$\begin{aligned} \dot{x}_1 &= (-x_2 - x_3)f_1 + x_1 f_2 \\ \dot{x}_2 &= x_1 f_1 + x_2 f_2 + f_3 \\ \dot{x}_3 &= x_1 f_1 + x_3 f_2 - f_3 \end{aligned} \tag{4.6}$$

Condition (3.6) yields

$$\begin{aligned} &(-x_1^2 - x_2 x_3 - x_2^2)f_2 + (-x_2 - x_3)f_3 = 0 \\ &(-x_1^2 - x_2 x_3 - x_3^2)f_2 + (x_2 + x_3)f_3 = 0 \\ &(-x_1 x_2 + x_1 x_3)f_2 - 2x_1 f_3 = 0 \end{aligned} \tag{4.7}$$

Note that the function f_1 does not occur here. It can easily be seen that the equations (4.7) can be fulfilled by inserting the critical points of X. Whether or not other solutions of the equations (4.7) exist depends on the form of the function f_2 and f_3. We put $\tilde{f}_2 = 1 - x_1^2 - x_2^2 - x_3^2$, $\tilde{f}_3 = x_2 - x_3$, and $f_1 = 1$. Moreover, we introduce two real parameters μ and ν and we set $f_2 = \mu\tilde{f}_2$

and $f_3 = \nu\tilde{f}_3(\mu, \nu \neq 0)$. Then we have the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 + \mu x_1(1 - x_1^2 - x_2^2 - x_3^2) \\ \dot{x}_2 &= x_1 + \mu x_2(1 - x_1^2 - x_2^2 - x_3^2) + \nu(x_2 - x_3) \\ \dot{x}_3 &= x_1 + \mu x_3(1 - x_1^2 - x_2^2 - x_3^2) - \nu(x_2 - x_3) \end{aligned} \tag{4.8}$$

Hence if $x_1^2 + x_2^2 + x_3^2 = 1$ and $x_2 = x_3$, then the equations (4.7) are also satisfied. Both equations define a surface and the intersection of them leads to a closed curve, i.e., a periodic solution of the systems (4.2) and (4.8). This periodic solution is the limit cycle of system (4.8). The curve can be given explicitly, namely,

$$x_1(t) = \cos t, \qquad x_2(t) = \sin t/2^{1/2}, \qquad x_3(t) = \sin t/2^{1/2} \tag{4.9}$$

We note in passing that Hopf's bifurcation theorem may be applied to equation (4.8) in order to prove that the system contains a periodic solution (Hopf, 1942).

Now we ask which role the real parameters μ and ν play. Obviously for certain choices of μ and ν other critical points also exist besides the critical point $x = 0$. We set $\mu = 1$. If $\nu \leqslant -\frac{1}{2}$, then only one real critical point exists. On the other hand, at $\nu \geqslant -\frac{1}{2}$ two other critical points exist, namely, $x_1 = 0$, $x_2 = \pm[(1 + 2\nu)/2]^{1/2}$, $x_3 = \pm[(1 + 2\nu)/2]^{1/2}$. The quantity μ is important for the stability of the system, i.e., the stability of the limit cycle. The stability of the limit cycle can be investigated via Liapunov's theory (Nemytskii and Stepanov, 1960).

5. *Conclusion*

In summary, we establish the following: For a given autonomous system $\dot{x} = X(x)$ and therefore a given vector field X we have derived a vector field Y with an associated autonomous system $\dot{x} = Y(x)$ such that equation (1.2) holds. In the two-dimensional case the methods for such a construction have been known for a long time. In the present paper we have considered the three-dimensional case and have constructed for a special vector field X a vector field Y such that equation (1.2) holds. Applying equation (1.2) we are able to obtain invariant surfaces and invariant curves of the vector field Y.

So far it is obvious that the construction of the vector field Y, when a vector field X is given, does not depend on whether or not the system $\dot{x} = Y(x)$ contains limit cycles. In order to consider this point we have to choose a special class of vector fields X. The basic roles are played by the invariants of X, say, $\gamma_1(x)$ and $\gamma_2(x)$. The only requirements are that $\gamma_1(x) = 0$ and $\gamma_2(x) = 0$ define surfaces and that the intersection of them lead to a closed curve.

Appendix

The given approach can also be considered from another point of view. Let $M = R^n$ (or an open subset of R^n). X is a vector field on M and α an r-form on $M (r \leqslant n)$. We call α a conformal invariant r-form of X if $L_X\alpha = g\alpha$. $L_X\alpha$ is the Lie derivative of α

with respect to X and g an arbitrary smooth function. For practical calculation one uses the rule $L_X\alpha = X \lrcorner (d\alpha) + d(X \lrcorner \alpha)$, where $X \lrcorner \alpha$ is the inner product of X and α.

The relation between (1.2) and the invariance requirement given above can be seen by the following theorem.

Theorem A.1. Let $M = R^n$ (or an open subset of R^n) and $\omega = dx_1 \wedge \ldots \wedge dx_n$. Let X and Y be two vector fields such that $[X, Y] = fY$ and $\alpha = Y \lrcorner \omega$. Then

$$L_X\alpha = (f + \mathrm{div}X)\alpha$$

Proof.
$$\begin{aligned} L_X(Y \lrcorner \omega) &= [X, Y] \lrcorner \omega + Y \lrcorner (L_X\omega) = \\ &= (fY) \lrcorner \omega + Y \lrcorner (X \lrcorner d\omega + d(X \lrcorner \omega)) = \\ &= f(Y \lrcorner \omega) + Y \lrcorner (d(X \lrcorner \omega)) = f(Y \lrcorner \omega) + Y \lrcorner (\mathrm{div}X\omega) = \\ &= (f + \mathrm{div}X)(Y \lrcorner \omega). \end{aligned}$$

Hence we have $g = f + \mathrm{div}X$.

There are the following properties. Let X be a vector field on M, and α, β conformal invariants with respect to X. Then $X \lrcorner \alpha$ and $\alpha \wedge \beta$ are conformal invariants with respect to X. If g is a constant, then $d\alpha$ is a conformal invariant. Finally, let α be a conformal invariant with respect to X and Y where g is a constant. Then α is a conformal invariant with respect to the vector field $[X, Y]$.

In the present approach we have the requirement

$$X \lrcorner (Y \lrcorner \omega) = 0$$

instead of equation (3.6).

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